Homotopy Transfer

Fix **homotopy data**

$$\begin{tikzcd} (A,d_A) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (H,d_H) \arrow[l, shift left, "i"] \end{tikzcd}$$

of (co)chain complexes, i.e.

- (A,d_A) and (H,d_H) are (co)chain complexes.
- i,p are chain maps.
- h is a homotopy $i \circ p \sim id_A$.

Question: What kind of structures on A can we transfer along such homotopy data?

Baby Example

Example ("perturbing the differential"): Let $\delta: A \to A$ be a map of same degree as the differential d_A s.t.

- $(d_A+\delta)^2 = 0$
- $(1-\delta h)$ is invertible.

Then we obtain new homotopy data

$$\begin{tikzcd} (A,d_A+\delta) \arrow[loop left, "h+h(1-\delta h)^{-1}\delta h"] \arrow[r, shift left, "p+p(1-\delta h)^{-1}\delta h"] & (H, d_H+p(1-\delta h)^{-1}\delta i) \arrow[l, shift left, "i+h(1-\delta h)^{-1}\delta i"] \end{tikzcd},$$

i.e. everything that is supposed to be a differential is a diff., everything that is supposed to be a chain map is a chain map etc...

Moreover, if i is a qis, then so is $i+h(1-\delta h)^{-1}\delta i$.

Proof: See calculations in [CRAINIC].

dgas

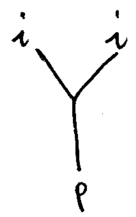
What if A is a (non-unital) dga? Can we transfer the multiplication to H s.t. it also becomes a dga?

Let $\nu : A \otimes A \to A$ be the mult. map.

Picture : Y.

Can define the "candidate multiplication" on H via

$$\begin{tikzcd} H \otimes H \arrow[d, "i \otimes i"] \arrow[ddd, phantom, shift left=8, "\mu_2"] \\ A \otimes A \arrow[d, "\nu"] \\ A \arrow[d, "p"] \\ H \end{tikzcd}$$



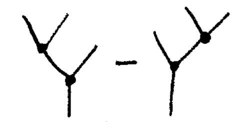
(shorthand: )

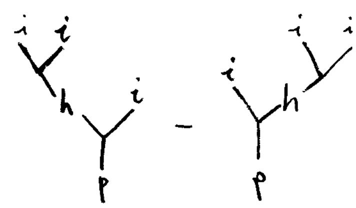
Now, for $x, y, z \in H$, there is no reason for

$$p(i(p(i(x)\cdot i(y)))\cdot i(z)) = p(i(x)\cdot i(p(i(y)\cdot i(z))))$$

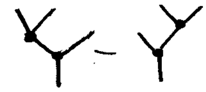
() ()

to hold if $i \circ p \neq \mathrm{id}$.

But:  is null homotopic (as a chain map $H^{\otimes 3} \to H$)! A null homotopy is given by

$\mu_3 :=$  $: H^{\otimes 3} \to H$, degree $+1$ (same degree as h).

$=$ .

<u>In fancier terms</u>: $d($  $) =$  in the complex $\mathcal{H}om(H^{\otimes 3}, H)$.

In fact, higher associativity relations are also encoded in certain boundaries in $\mathcal{H}om$ complexes...

Interlude on A_∞-algebras

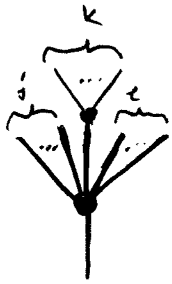
Def.: An A_∞-algebra is a chain complex X equipped with maps $\gamma_n : X^{\otimes n} \to X$ of degree $n-2$ for $n \geq 2$ such that

Picture:

$$d(\gamma_n) = -\sum_{\substack{1<k<n \\ j+k+l=n}} (-1)^{jk+l} \gamma_{n-k+1} \circ (\underbrace{id \otimes \dots \otimes id}_{j \text{ factors}} \otimes \gamma_k \otimes \underbrace{id \otimes \dots \otimes id}_{l \text{ factors}})$$

differential of $\mathcal{H}om(X^{\otimes n}, X)$

$$= -\sum_{\substack{1<k<n \\ j+k+l=n}} (-1)^{jk+l}$$



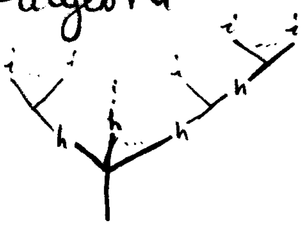
for all $n \geq 2$.

Prop.: dgas correspond to A_∞-algebras $(X, \gamma_\bullet)$ s.t. $\gamma_n = 0$ f.a. $n \geq 3$.

Back to Transferring Structure

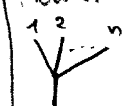
Prop.: Assume (in the homotopy data) that A is equipped with an A_∞-algebra structure $(\gamma_n)_{n\geq 2}$. Then H has an A_∞-algebra structure given by

$$\mu_n = \sum_{\substack{\text{planar} \\ \text{rooted} \\ \text{trees} \\ \text{with} \\ n \text{ leaves}}} \pm$$



$(\star)$.

Picture:



In particular, this construction yields an A_∞-alg. structure on H if A is a dga.

Remark: If A is a dga, the sum $(\star)$ simplifies to a sum over planar rooted **binary** trees.

④

<u>Example</u>: Let A be a dga such that there are splittings $A_n \cong \ker(d_n) \oplus \mathrm{im}(d_n) = Z_n \oplus B_{n-1}$
and $Z_n \cong H_n(A) \oplus \ker(d_{n-1}) = H_n(A) \oplus B_n$. (★★)

Then we obtain inclusions $i_n : H_n(A) \hookrightarrow A_n$,
projections $p_n : A_n \twoheadrightarrow H_n(A)$,
and $h_n : A_n \twoheadrightarrow B_n \hookrightarrow A_{n+1}$,

which assemble into homotopy data

$$\begin{tikzcd} (A,d) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (H_*(A),0) \arrow[l, shift left, "i"] \end{tikzcd}.$$

Thus we can transfer the dga structure on A to an A_∞-algebra structure $(\mu_n : H_*(A)^{\otimes n} \to H_*(A))_n$ on $H_*(A)$.

Even though μ_2 is associative (for the "trivial" reason that the diff. of $H_*(A)$ is zero), the higher A_∞-multiplications μ_n, $n \geq 3$ may be non-trivial (more on that later).

<u>Special case</u>: Let X be a space, k a field. Set A to be $C^*_{sing}(X;k)$ equipped with the cup product. Then we can find splittings as in (★★) and in the transferred structure μ_2 corresponds to the cup product in cohomology.

<u>More geometric/analytic example</u>: Let M be a cpl. oriented mfd. Then we have

<u>Thm.</u> (Hodge): The choice of a Riemannian metric g on M yields an isomorphism

$$\Omega^*(M) \supseteq \mathcal{H}^*_g(M) \xrightarrow{\cong} H^*_{dR}(M)$$

"harmonic forms w.r.t. g"

and homotopy data

$$G \circlearrowright (\Omega^*(M), d) \underset{i}{\overset{p}{\rightleftarrows}} (H^*_{dR}(M), 0).$$

The transfer of the wedge product on $\Omega^*(M)$ along this data coincides with the wedge product on $H^*_{dR}(M)$ (which corresponds to the cup product under the iso. $H^*_{dR}(M) \xrightarrow{\cong} H^*(M;\mathbb{R})$)

Now back to higher multiplications:

Massey Products

Def.: For a dga A with the required splittings, we call the multiplication maps $(M_n : H_*(A)^{\otimes n} \to H_*(A))_n$ A_∞-Massey products.

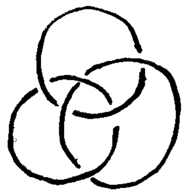
Def. ("classical" Massey products): Let A be a dga, let $[u],[v],[w] \in H_*(A)$ be homogeneous elements st $[u]\cdot[v] = 0 = [v]\cdot[w]$. Then their triple Massey product is defined as

$$\langle [u],[v],[w] \rangle = \left\{ [(-1)^{|u|} u \cdot x + (-1)^{|y|} y \cdot w] \;\middle|\; dx = (-1)^{|v|} v \cdot w,\ dy = (-1)^{|u|} u \cdot v \right\}$$

$$\in H_*(A) / ([u] \cdot H_*(A) + H_*(A) \cdot [w]).$$

Prop.: In the setup of the previous definitions, we have $M_3([u],[v],[w]) \in \langle [u],[v],[w] \rangle$.

Example: "Borromean Rings"



(Pairwise trivial linking number, but non-trivial triple Massey product in $S^3 \setminus \{⊛\}$.)

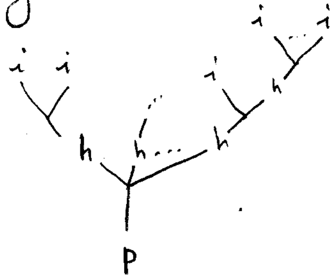
L_∞-algebras

We can play the same game with L_∞ algebras:

Prop.: Assume that A is equipped with the structure of an L_∞ algebra with "brackets" $(\ell_n^A : A^{\otimes n} \to A)_{n \geq 1}$. Then this yields an L_∞-algebra structure on H via

Picture:

$$\ell_n^H = \sum_{\substack{\text{rooted} \\ \text{trees}}} \pm$$



for $n \geq 2$.

In particular, dgla structures can be transferred to L_∞ structures.

Example: V dgla over a field $\underset{\text{choose splittings}}{\rightsquigarrow}$ $(H^*(V), 0)$ L_∞-alg.

Further Reading

Many aspects of L_∞-, A_∞- etc. algebras can be unified and conceptualized with the theory of operads. For example, a more general "homotopy transfer theorem" can be shown for so-called Koszul operads. See [VALLETTE] for an account of operads.

References

- [CRAINIC]: Crainic, Marius. On the Perturbation Lemma, and Deformations
- [VALLETTE]: Vallette, Bruno. Algebra + Homotopy = Operad.